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ON THE TRANSITION FROM HEAVY TRAFFIC TO HEAVY TAILS FOR THE M/G/1 QUEUE: THE REGULARLY VARYING CASE

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1. Extended proof of Lemma 3.2. This is an extended proof of Lemma 3.2 from Olvera-Cravioto et al. (2009) that includes the case when $\alpha = 3$. The value $\alpha = 3$ constitutes the boundary between infinite and finite variance, and results about the asymptotic behavior of $P(S_n > x)$ usually imply additional technical subtleties. For this reason most authors have ignored this specific value of α.

LEMMA 3.2. *Let $X_1, X_2, \ldots$ be iid nonnegative random variables with $\mu = E[X] < \infty$, and $P(X_1 > t) = t^{-\alpha+1}L(t)$ where $L(\cdot)$ is slowly varying and $\alpha > 2$. Set $S_n = X_1 + \cdots + X_n$, $n \geq 1$. For any $(2 \wedge (\alpha-1))^{-1} < \gamma < 1$ define $M_\gamma(x) = \lfloor (x - x^\gamma)/\mu \rfloor$. Then, there exists a function $\varphi(t) \downarrow 0$ as $t \uparrow \infty$ such that*

$$\sup_{1 \leq n \leq M_\gamma(x)} \left| \frac{P(S_n > x)}{nP(X_1 > x - (n-1)\mu)} - 1 \right| \leq \varphi(x).$$

PROOF. Suppose first that $\alpha > 3$ and let $\sigma(n) = \sqrt{(\alpha-2)n \log n}$. Since

$$\frac{P(S_n > x)}{nP(X_1 > x - (n-1)\mu)} = \frac{P(S_n^* > x - n\mu)}{nP(Y_1 > x - n\mu)},$$

where $Y_i = X_i - \mu$ and $S_n^* = Y_1 + \cdots + Y_n$. Then the result will follow from Theorem 4.4.1 from Borovkov and Borovkov (2008) once we show that $(x - n\mu)/\sigma(n) \to \infty$ uniformly for $1 \leq n \leq M_\gamma(x)$. To see this simply note that

$$\frac{x - n\mu}{\sigma(n)} \geq \frac{x - M_\gamma(x)\mu}{\sigma(M_\gamma(x))} \sim \sqrt{\frac{\mu}{\alpha-2}} \cdot \frac{x^{\gamma-1/2}}{\sqrt{\log x}}.$$

Since $\gamma > 1/2$, the above converges to infinity.

Suppose now that $\alpha \in (2,3)$ and note that $P(Y_1 \leq -t) = 0$ for $t \geq \mu$. Note also that since $\overline{F}(t) = P(Y_1 > t)$ is regularly varying with index $\alpha - 1$, then $\sigma(n) = \overline{F}^{-1}(1/n) = n^{1/(\alpha-1)}\tilde{L}(n)$ for some slowly varying function $\tilde{L}(\cdot)$ (see Bingham et al., 1987). Then the result will follow from Theorem 3.4.1 from Borovkov and Borovkov (2008) once we show that $(x - n\mu)/\sigma(n) \to \infty$ uniformly for $1 \leq n \leq M_\gamma(x)$. To see this note that

$$\frac{x - n\mu}{\sigma(n)} \geq \frac{x - M_\gamma(x)\mu}{\sigma(M_\gamma(x))} \sim \frac{x^\gamma}{\sigma(x/\mu)} \sim \frac{x^{\gamma - 1/(\alpha-1)}}{\mu^{-1/(\alpha-1)}\tilde{L}(x)},$$

and since $\gamma > 1/(\alpha - 1)$, the above converges to infinity.

We now give the proof for the case $\alpha = 3$; the arguments we give here are based on an upper and lower bound. Let $1/2 < \eta < \gamma$ and $y = x - n\mu - x^\eta$. Define

$$V_\alpha(t) = \begin{cases} \frac{1}{t^2}\int_0^t uP(Y_1 > u)\,du, & \text{if } \int_0^\infty uP(Y_1 > u)\,du = \infty, \\ \frac{1}{t^2}\int_0^\infty uP(Y_1 > u)\,du, & \text{if } \int_0^\infty uP(Y_1 > u)\,du < \infty, \end{cases}$$

and

$$W_\beta(t) = \frac{1}{t^2}\int_0^\infty uP(Y_1 < -u)du.$$

Set

$$\Pi^* = n\left[V_\alpha\left(\frac{y}{|\ln(nP(Y_1 > x - n\mu))|}\right) + W_\beta\left(\frac{y}{|\ln(nP(Y_1 > x - n\mu))|}\right)\right].$$

Note that for $1 \leq n \leq M_\gamma(x)$,

$$\frac{y}{|\ln(nP(Y_1 > x - n\mu))|} \geq \frac{y}{|\ln P(Y_1 > x - \mu)|} \sim \frac{x - n\mu}{(\alpha - 1)\ln x}.$$

Therefore,

$$\begin{aligned} \sup_{1 \leq n \leq M_\gamma(x)} \Pi^* &\leq \sup_{1 \leq n \leq M_\gamma(x)} Cn\left[V_\alpha\left(\frac{x - n\mu}{\ln x}\right) + W_\beta\left(\frac{x - n\mu}{\ln x}\right)\right] \\ &\leq C\frac{x}{\mu}\left[V_\alpha\left(\frac{x^\gamma}{\ln x}\right) + W_\beta\left(\frac{x^\gamma}{\ln x}\right)\right] \\ &\sim C\frac{x}{\mu} \cdot x^{-2\gamma}\tilde{L}(x) \\ &\leq C' x^{-2\eta+1} \end{aligned}$$

for some constants $C, C' > 0$ and some slowly varying function $\tilde{L}$. Since $2\eta - 1 > 0$, then the above converges to zero, and by Corollary 3.1.7 from

Borovkov and Borovkov (2008),

$$\sup_{1\leq n\leq M_\gamma(x)} \frac{P(S_n^* > x - n\mu)}{nP(Y_1 > x - n\mu)} \leq 1 + \epsilon(x^{-2\eta+1}),$$

for some $\epsilon(t) \downarrow 0$ as $t \downarrow 0$.

For the lower bound redefine $y = x - n\mu + x^\beta\sqrt{n-1}$, $\beta = \eta - 1/2$, and let $Q_n(u) = P(S_n^*/\sqrt{n} < -u)$; note that $y \sim x - n\mu$ as $x \to \infty$, uniformly for $1 \leq n \leq M_\gamma(x)$. By Theorem 2.5.1 from Borovkov and Borovkov (2008) we have

$$\begin{aligned}
&\inf_{1\leq n\leq M_\gamma(x)} \frac{P(S_n^* > x - n\mu)}{nP(Y_1 > x - n\mu)} \\
&\geq \inf_{1\leq n\leq M_\gamma(x)} \frac{P(Y_1 > y)}{P(Y_1 > x - n\mu)} \left(1 - Q_{n-1}(x^\beta) - \frac{n-1}{2}P(Y_1 > y)\right) \\
&\geq C \inf_{1\leq n\leq M_\gamma(x)} \left(1 - Q_{n-1}(x^\beta) - nP(Y_1 > y)\right)
\end{aligned}$$

for some constant $C > 0$. We will prove that the expression above converges to one. We start by noting that

$$\begin{aligned}
\sup_{1\leq n\leq M_\gamma(x)} nP(Y_1 > y) &\leq M_\gamma(x)P(Y_1 > x - M_\gamma(x)\mu) \\
&\leq \frac{x}{\mu}P(Y_1 > x^\gamma) \\
&\sim \frac{x^{1-(\alpha-1)\gamma}}{\mu}L(x^\gamma).
\end{aligned}$$

Since $(\alpha - 1)\gamma - 1 > 0$, then the above converges to zero. Finally, choose $1 < 1/\eta < \kappa < 2$. Then, by Pyke and Root (1968), $E[|\hat{Z}_n|^\kappa] = o(n)$ as $n \to \infty$, so there exists a constant $C' > 0$ such that

$$Q_{n-1}(x^\beta) = P(-S_{n-1}^* > x^\beta\sqrt{n-1}) \leq \frac{E\left[|S_{n-1}^*|^\kappa\right]}{x^{\beta\kappa}(n-1)^{\kappa/2}} \leq \frac{C'(n-1)^{1-\kappa/2}}{x^{\beta\kappa}}.$$

It follows that

$$\sup_{1\leq n\leq M_\gamma(x)} Q_{n-1}(x^\beta) \leq \frac{C'x^{1-\kappa/2-\beta\kappa}}{\mu^{1-\kappa/2}}.$$

Our choice of κ guarantees that $1 - \kappa/2 - \beta\kappa = 1 - \kappa\eta < 0$, so the above converges to zero. This completes the proof. □

References.

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